Architecture Image Studies

Vol.7, Issue 1, pp.2363-2368, 2026
DOI: https://doi.org/10.62754/ais.v7i1.1223

https://journals.ap2.pt/index.php/ais/index

Bifurcation Analysis for Some Fractional SIRS Models

Mahmoud M. El-Borai[1], Wagdy G. El-Sayed[2], Ahmed M. Ibrahim[3], Jawaher Th. Alhamdan[4]

Abstract

In this paper , some fractional-order epidemic models are investigated. In particular , a fractional SIRS epidemic model is considered to describe the transmission dynamics of infectious diseases incorporating memory effects through fractional derivatives . The existence and stability of equilibrium points of the proposed model are analyzed .Moreover, analytical solution are obtained for some special cases , highlighting the impact of fractional-order dynamics on the behavior of the epidemic system.

Keywords: *Dynamical Systems, Fractional Epidemic Models, Stability, Mittag-Lefler Function.*

Introduction

Fractional-order differential equations have received considerable attention in recent years due to their ability to model memory and hereditary properties inherent in many real-world phenomena , including epidemiological processes In contrast to classical integer-order models , fractional epidemic models provide a more accurate description of disease transmission dynamics .

In the following sections we shall study some fractional epidemic models.

The Fractional SIRS Epidemic Model

Let $S(t)\,.\,I(t)\,.\,and\ R(t)$ denote the numbers of susceptible infected , and recovered individuals at time t , respectively . The fractional SIRS model is given by :

$$\begin{cases} \frac{d^{\alpha}S(t)}{dt^{\alpha}} = b - cS(t) - f(I)S^{*} + \delta R(t), \\ \frac{d^{\alpha}I(t)}{dt^{\alpha}} = f(I)S^{*} - (c+\mu)I(t), \\ \frac{d^{\alpha}R(t)}{dt^{\alpha}} = \mu\, I(t) - (c+\delta)R(t), \end{cases} \qquad (2\cdot 1)$$

$$S^{*}(t) = \lim_{t\to\infty} N(t) - I(t) - R(t),$$

, see [1-5]. Where $0 < \alpha \le 1$. $b > 0$

The total population size is defined as :

$$N(t) = S(t) + I(t) + R(t)$$

Description of Parameters

The parameters appearing in system (1.1) are defined as follows :

[1] Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, 21511 Alexandria, Egypt, Email: m_m_elborai@yahoo.com
[2] Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, 21511 Alexandria, Egypt, Email: Wagdygoma@alexu.edu.eg.
[3] Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, 21511 Alexandria, Egypt, Email: Ahmedsayed2890@gmail.com
[4] Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, 21511 Alexandria, Egypt, Email: Alhamdanjawaher45@gmail.com , (Corresponding Author)

$b > 0$: the recruitment rate of the population

$c > 0$: the natural death rate

$\mu > 0$: the recovery rate of infected individuals

$\delta > 0$: the rate at which recovered individuals lose immunity and return to the susceptible class

$f(I)$: the incidence rate function describing the interaction between susceptible and infected individuals .

The incidence rate is assumed to be of the nonlinear form :

$$f(I)S = \frac{BI}{I + VI^k} S$$

Where $B > O . V > 0$ $and\ k > 0$ $are\ constants$

Let $S(t) . I(t) . and\ R(t)$ denote the numbers of susceptible , infective, and removed individuals at time t , respectively .

The parameter β represents the average number of new infections generated per unit time in a fully susceptible population .

For convenience , we introduce the following notations :

$$\lambda = \frac{b}{a} \quad . \quad \Lambda_0 = \frac{\beta}{C + \delta} \Lambda \quad . \quad R_0 = \frac{\Lambda_0}{\delta}$$

Where all parameters are assumed to be positive constants.

This work is organized as follows . In Section 1 , we investigate the fractional-order system given by equation (1.1) . In Section 2 , we analyze several epidemic models , highlighting their qualitative behavior and mathematical properties .

4. Population Dynamics

By summing the equation of system (1.1) , we obtain the fractional differential equation governing the total population :

$$\frac{d^\alpha N(t)}{dt^\alpha} = b - c\, N(t) \qquad (4 \cdot 1)$$

This equation implies that the total population approaches the steady-state value $N^* = \frac{b}{c}$ $as\ t \to \infty$. ensuring the biological feasibility of the model .

Where $N(t) = S(t) + I(t) + R(t)$

Using the result in [6] , we can obtain the solution of (4.1) in the form:

$$N(t) = \int_0^\infty \xi_\alpha(\theta) e^{-c(t-\theta)^\alpha} d\theta + N(O) + \alpha \int_0^t \int_0^\infty \theta b \xi_\alpha(\theta)(t-y)^{\alpha-1} e^{-c(t-y)^\alpha} d\theta dy$$

Consequently

$$N(t) = \frac{b}{c} + \left(N(0) - \frac{b}{c} \right) E_\alpha \left(-ct^\alpha \right) . \ t \geq 0 \quad . \ (4.2)$$

Type equation here.

Where

$$E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + 1)}. \quad \alpha > 0$$

Is the Mittag-Leffler function

So

$$\lim_{t \to \infty} N(t) = \frac{b}{c}$$

Now we can write

$$\begin{cases} \frac{d^\alpha I(t)}{dt^\alpha} = f(t)[\beta N(t) - R(t) - I(t)] - (c + \mu) I(t). \\ \frac{d^\alpha R(t)}{dt^\alpha} = \mu I(t) - (c + \delta) R(t) \qquad (4.3) \end{cases}$$

System (1.5) can be written in the form

$$\begin{cases} \frac{d^\alpha x(t)}{dt^\alpha} = \alpha(1 + \rho x^{k-1})(N^*(t) - x - y)x(t) - \gamma x \\ \frac{d^\alpha y(t)}{dt^\alpha} = \gamma x - y \qquad (4.4) \end{cases}$$

Where the dimensionless variables and parameters are defined by

$$I = \frac{c + \delta}{\beta} x \ . \qquad R = \frac{C + \delta}{\beta} y \ . \qquad t = \frac{1}{c + \delta} \tau \ .$$

$$P = \left(\frac{C + \delta}{\beta}\right)^{k-1} . \quad N^* = \frac{\beta}{c + \delta} N \ .$$

$$\gamma = \frac{C + M}{C + \delta} \ . \quad \eta = \frac{\mu}{C + \delta}$$

According to the results in $[1 - 3]$

We can deduce that (0.0) is a unique disease free equilibrium for system (4.4)

The disease free equilibrium $(0.0)$is a stable node if

$$R_0 < 1 \ . \ R_0 > 1$$

If $R_0 = 1$. then (0.0) is a saddle node .

2. Some fractional modified SIR models

Consider the following epidemic fractional epidemic model

$$\frac{d^\alpha S}{dt^\alpha} = -bSI - aS$$

$$\frac{d^\alpha I}{dt^\alpha} = bSI - aI$$

$$\frac{d^\alpha R}{dt^\alpha} = a(I + S)$$

Let us suppose that the initial conditions

$$S(0) < 1 \ . \ I(0) > 0 \ .$$

$$R(0) = 0 \ \ and$$

$$S(0) + I(0) = 1$$

We have

$$\frac{d^\alpha N(t)}{dt^\alpha} = 0$$

Thus

$$N(t) = S(0) + I(0) + R(0) = 1$$

$$\frac{d^\alpha R}{dt^\alpha} = a[1 - R]$$

$$= a - aR$$

Using the results in [6] , we can get

$$R(t) = a \int_0^t \int_0^\infty \alpha\theta(t-\eta)^{\alpha-1} e^{-a(t-\eta)^\alpha \theta} \xi_\alpha \, d\theta \, dy$$

$$= 1 - \mathrm{E}_\alpha(-at^\alpha) \, .$$

Where

$$E_\alpha(-at^\alpha) = \sum_{j=0}^{\infty} \frac{(-at^\alpha)^j}{\Gamma(\alpha j + 1)} . \qquad \alpha > 0$$

Is the Mittag-Leffler function .

Thus

It is clear that

$$\lim_{t\to\infty} R(t) = 1$$

$$S(t) + I(t) = E_\alpha(-at^\alpha) \, .$$

It follows that

$$\lim_{t\to\infty} S(t) = \lim_{t\to\infty} I(t) = 0$$

3. The fractional SIR model

Consider

$$\begin{cases} \frac{d^\alpha S}{dt^\alpha} = -bSI \\ \frac{d^\alpha I}{dt^\alpha} = bSI - aI \quad . \quad (3 \cdot 1) \\ \frac{d^\alpha R}{dt^\alpha} = aI \end{cases}$$

$$S > O . \quad I > 0 \ and \ R \geq 0 \ .$$

$$for \ all \ t \geq 0 \ . \ o < b < a$$

$$S(0) + I(0) = 1$$

$$R(0) = O$$

$$S(t) + I(t) + R(t) = 1$$

Thus

$$\frac{d^\alpha R(t)}{dt^\alpha} \leq a - aR(t).$$

So

$$R(t) \leq a \int_0^t \int_0^\infty \alpha\theta(t-\eta)^{\alpha-1} \mathrm{e}^{-a(t-\eta)^\alpha \theta} \xi_\alpha \, d\theta \, dy$$

$$R(t) \leq 1 - E_\alpha(at^\alpha)$$

Thus the proportion of removal $\mathcal{R}$ starts at zero and steadily increases , eventually approaching .

As a limit

$$\lim_{t\to\infty} R(t) \leq 1$$

Notice that

$$\frac{d^\alpha I}{dt^\alpha} = b\,[1 - R - I]\, I - aI$$

$$\leq -(a-b)I$$

Thus

$$I(t) \leq I(0) \int_0^\infty \xi_\alpha(\theta) \mathrm{e}^{-(a-b)(t-\theta)^\alpha} d\theta$$

Consequently

$$\lim_{t\to\infty} I(t) = 0$$

This means that the proportion $I(t)$ of infected approaches zero as t tends to ∞

We notice that

$$\frac{d^\alpha S(t)}{dt^\alpha} > -bS(t)$$

Thus

$$S(t) > S(0) \int_0^\infty \xi_\alpha(\theta) \mathrm{e}^{-bt^\alpha} d\theta$$

This means that the proportion of susceptible $S(t)$ starts at $S(0)$ and steadily decreases , eventually approaching a limit

$\lim_{t\to\infty} S(t) > 0$, see [7-12].

Conclusion

Some fractional epidemic models are studied. Analytical solutions are obtained for some special cases. The stability of solutions are obtained.

References

[1] Xiao ling Wang and Kuilin Wu , Bifurcation analysis for a SIRS model with a nonlinear incidence rate, arXiv :2510.2729, VI (math.DS), 31 Oct 2025 pp. 1-25.

[2] William G.Faris, The SIR model: an epidemic model, arXiv:2104.12029v1 . [math.DS] 24 Apr2021, pp. 1-15.

[3] Tiberiu Harko, Francisco S. N. , Lobo and M. K. Mak, Exact analytic solutions of the Susceptible-Infected-Recovered (SIR) epidemic model and of the SIR model with equal death and birth rates, Applied Mathematics and Computation, 236 (2014), 184-194.

[4] W.O.Kermack and A. G. Mckendrick, A contribution to the mathematical theory of epidemics, Proceedings of the Royal Society A, 115(1927), 700-721.

[5] Roman Cherniha and Vasyl Davydovych, A mathematical model for the Covid-19 outbreaks and its applications, Symmetry 2020, 12, 990, 1-19

[6] Mahmoud M. El-Borai , Some probability densities and fundamental solution of fractional evolution equations , Chaos, Soliton and Fractals 14 (2002),433-440

[7] M.M. El-Borai and K. E.-S. El-Nadi, " On the construction of a fractional normal distribution and a related fractional Brownian motion," Glod. Sci. J., vol. 12, no. 3, pp. 1125-1131, Mar. 2024.

[8] M.M. El-Borai, W. G. El-Sayed, and R. M. Al-Masroud, "Exact solutions for time fractional coupled Whitham-Broer-Kaup equations via exp-function method," 2015.

[9] W. G. El-Sayed and A. M. El-Sayed, " On the functional integral equations of mixed type and integro-differential equations of fractional orders, " Appl. Math. Comput., vol. 154, no. 2, pp. 461-467 , July 2004.

Doi: 10. 1016/S0096-3003(03)00727-6.

[10] K. E.-S. El-Nadi, M. A. Abdou, and M. A. Fahmy, "Fractional Poisson and Gamma Models in Rainfall Analysis: Impact on Climate Change and Marine Ecosystems," Math. Comput. Sci. Res. Updat. Vol 1, pp.21-59, Jan. 2025, doi: 10.9734/bpi/mcsru/v1/3243.

[11] M.M. El-Borai and K. E.-S. El-Nadi, " On Some Stochastic Parabolic Systems Driven by New Fractional Brownian Motions," Asian J. Probab. Stat., vol. 26, no. 9, pp. 1-8, Aug. 2024, doi: 10.9734/ajpas/2024/v26i9642.

[12] Mahmoud M. El-Borai, Wagdy A. El-Sayed, Mahmoud A. Habib, Plankton-oxygen dynamics in the context of climate change, A fractional model with probability density function approach, Architecture Image Studies Science Journal, vol. 6, Issue 3, 2025, pp. 1-5.